1. The joint density of \((X, Y, Z)\) is given by

\[
f(x, y, z) = \begin{cases} 
\frac{1 - \sin x \sin y \sin z}{8\pi^3}, & 0 \leq x, y, z \leq 2\pi; \\
0, & \text{otherwise.}
\end{cases}
\]

(a) Find the density for \((X, Y)\).

(b) Find the density for \(X\).

(c) Prove \(X, Y, Z\) are pairwise independent.

(d) Are \(X, Y, Z\) independent?

2. Let \(X_1, \ldots, X_n, Y_1, \ldots, Y_n\) be IID from Bernoulli\(\frac{1}{2}\). Define \(Z_i = X_i - Y_i\) for \(i = 1, \ldots, n\).

(a) Find the probability mass function of \(Z_1\).

(b) Find \(E[\left(\sum_{i=1}^{n} Z_i\right)^2]\).

(c) Find \(E[\left(\sum_{i=1}^{n} Z_i\right)^4]\).

3. Suppose that \(X_1, \ldots, X_n, Y_1, \ldots, Y_m\) are independent random variables. \(X_i \sim \text{Uniform}[0, \theta_1]\), \(i = 1, \ldots, n\), and \(Y_j \sim \text{Uniform}[0, \theta_2]\), \(j = 1, \ldots, m\).

(a) Find the joint density of order statistics \(f_{X(n), Y(m)}(u, v)\), where \(X_{(n)}\) and \(Y_{(m)}\) are the maximum observations for each sample respectively.

(b) Assume \(\theta_1 = \theta_2 = \theta\). Find the density of \(Z = \max\{X_{(n)}, Y_{(m)}\}\).

(c) If \(\theta_1 = \theta_2 = \theta\), define \(T = \frac{X_{(n)} Y_{(m)}}{Z_{(m+n)}}\). Find \(P(T < c)\) for \(0 < c < 1\). What is the distribution of \(T\)?

4. Let \(X_1\) and \(X_2\) be independent and identically distributed (iid) Bernoulli observations with \(p = P(X_1 = 1) = 1 - P(X_1 = 0)\), where \(0 \leq p \leq 1\) is unknown. It is desired to estimate \(\theta = P(X_1 = X_2)\).

(a) Specify the range of possible values for \(\theta\).

(b) Find an unbiased estimator of \(\theta\).

(c) Find the uniformly minimum-variance unbiased estimator (UMVUE) for \(\theta\). Justify the answer.

(d) Is the UMVUE obtained in (c) a reasonable estimator? Justify the answer.

(e) Find the maximum likelihood estimator (MLE) of \(\theta\).
5. Let $X_1, \ldots, X_n$ be a random sample with probability density function (pdf)

$$f(x; \theta) = \frac{2}{\sqrt{\pi \theta}} \exp\{-\frac{x^2}{\pi}\}, \quad x > 0,$$

where $\theta > 0$. (Hint: You may use the following facts: $\frac{2X^2}{\theta} \sim \chi^2_1$, a chi-square variable with 1 degree of freedom.)

(a) Find the Cramér-Rao Lower bound for estimating $\theta$.

(b) Using the Cramér-Rao Lower bound, find the UMVUE of $\theta$.

(c) Given $\theta_0 > 0$ and test size $\alpha \in (0, 1)$, show that the likelihood ratio test for

$$H_0 : \theta = \theta_0 \text{ vs } H_1 : \theta \neq \theta_0$$

is given by

Reject $H_0$ if $W_n < \theta_0 c_1$ or $W_n > \theta_0 c_2$,

where $0 < c_1 < c_2 < \infty$ are appropriate constants and $W_n = 2 \sum_{i=1}^n X_i^2$. Use chi-square quantiles to describe the required form of $c_1$ and $c_2$.

(d) Use the likelihood ratio test in (c) to find a confidence interval for $\theta$ with confidence coefficient $1 - \alpha$. Justify your solution.

6. Let $(X_i, Y_i), i = 1, \ldots, n$ be a random sample from a (bivariate) uniform distribution defined on a circle centered at $(0, 0)$ with radius $\theta > 0$. That is, the joint probability density function of $(X_1, Y_1)$ is given by

$$f(x, y; \theta) = \begin{cases} \frac{1}{\pi \theta^2} & \text{if } \sqrt{x^2 + y^2} < \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Define $R_n = \max_{i=1,\ldots,n} \sqrt{X_i^2 + Y_i^2}$. (Hint: if needed, you may use the following fact: $P(X_i^2 + Y_i^2 \leq s^2) = (s/\theta)^2$ for $s \in [0, \theta]$.)

(a) For $0 \leq r \leq \theta$, show that $P(\theta)(R_n \leq r) = (r/\theta)^{2n}$.

(b) Show that $R_n$ is a consistent estimator of $\theta$.

(c) Suppose a confidence interval for $\theta$ of the form $I_n = (0, R_n 5^{1/(2n+\sqrt{n})}]$. For a given $\theta > 0$, determine the coverage probability of $I_n$ as $n \to \infty$. 
Solutions:

1. (a) Density of \((X,Y)\) is \(\int_0^{2\pi} f(x, y, z)dz = \frac{1}{4\pi}\).

(b) Density of \(X\) is \(\int_0^{2\pi} \int_0^{2\pi} f(x, y, z)dydz = \frac{1}{2\pi}\).

(c) By last two questions, \(X\) and \(Y\) are independent. Same for other pairs.

(d) If \(X, Y, Z\) are independent, by (2), the joint density would be \(\frac{1}{8\pi}\). So they are not independent.

2. (a) \(P(Z_i = 0) = \frac{1}{2}, P(Z_i = 1) = P(Z_i = -1) = \frac{1}{4}\).

(b) \(E(Z_i^2) = \frac{1}{2}, E(Z_i Z_j) = E(Z_i) E(Z_j) = 0, i \neq j\). So \(E(\sum Z_i)^2 = E \sum Z_i^2 = \frac{n}{2}\).

(c) Write \((\sum Z_i)^4\) as a sum of degree 4 monomials. Only the monomials of type \(Z_i^4\) and \(Z_i^2 Z_j^2\) have nonzero expectation. \(E(Z_i^4) = \frac{1}{2}\) and \(E(Z_i^2 Z_j^2) = \frac{1}{4}, i \neq j\). There are \(n\) terms of type \(Z_i^4\) and \(\binom{n}{2} \binom{4}{2} = 3n(n-1)\) terms of type \(Z_i^2 Z_j^2\). So \(E(\sum Z_i)^4 = \frac{n}{2} + \frac{3n(n-1)}{4}\).

3. (a) \(P(X_{(n)} \leq u) = P(X_i \leq u, i = 1, \ldots, n) = (u/\theta_1)^n\), so the density of \(X_{(n)}\) is \(nu^{n-1}/\theta_1^n\). Similarly, \(Y_{(m)}\) has density \(mv^{m-1}/\theta_2^m\). As they are independent, the joint density is \(n\theta_1^{n-1} m\theta_2^{m-1}/\theta^{n+m}\).

(b) Given \(\theta_1 = \theta_2 = \theta\), \(X_i\)'s and \(Y_j\)'s are IID from \(\text{Uniform}[0, \theta]\), so \(Z\) has density \((n + m)z^{n+m-1}/\theta^{n+m}\).

(c) Note that \(T = \begin{cases} X_{(n)}/Y_{(m)}, & \text{when } X_{(n)} < Y_{(m)}; \\ Y_{(m)}/X_{(n)}, & \text{when } X_{(n)} > Y_{(m)}. \end{cases}\) We have

\[
P(T < c) = P(T < c, X_{(n)} < Y_{(m)}) + P(T < c, X_{(n)} > Y_{(m)})
= P(X_{(n)}/Y_{(m)} < c^{\frac{1}{n}}, X_{(n)} < Y_{(m)}) + P(Y_{(m)}/X_{(n)} < c^{\frac{1}{m}}, X_{(n)} > Y_{(m)})
= P(X_{(n)}/Y_{(m)} < c^{\frac{1}{n}}) + P(Y_{(m)}/X_{(n)} < c^{\frac{1}{m}})
\]

where the probabilities in the last line above are independent of the value of \(\theta\). So we can assume \(\theta = 1\), and

\[
P(X_{(n)}/Y_{(m)} < c^{\frac{1}{n}}) = \int_0^1 \int_0^{vc^{\frac{1}{n}}} nu^{n-1}mv^{m-1}dudv
= \int_0^1 (vc^{\frac{1}{n}})^n mv^{m-1}dv
= \int_0^1 cmv^{n+m-1}dv
= c\frac{m}{m+n}
\]

Similarly, the second summand equals \(c\frac{n}{m+n}\) and \(P(T < c) = c\) for all \(c \in (0, 1)\). That is, \(T \sim \text{Uniform}(0, 1)\).
4. Let $X_1$ and $X_2$ be independent and identically distributed (iid) Bernoulli observations with $p = P(X_1 = 1) = 1 - P(X_1 = 0)$, where $0 \leq p \leq 1$ is unknown. It is desired to estimate $\theta = P(X_1 = X_2)$.

(a) $\theta = P(X_1 = X_2) = P(X_1 = 0, X_2 = 0) + P(X_1 = 1, X_2 = 1) = p^2 + (1 - p)^2 \in [\frac{1}{2}, 1]$ for $p \in [0, 1]$. The parameter $\theta$ has a minimum value $\frac{1}{2}$ when $p = \frac{1}{2}$.

(b) The unbiased estimator of $\theta$ is $\hat{\theta} = I(X_1 = X_2)$.

(c) Show $S = X_1 + X_2$ is a complete and sufficient statistic. Then $\hat{\theta} = I(X_1 = X_2) = I(S \neq 1)$, is a function of $S$. By Lehmann-Scheffé theorem, $\hat{\theta}$ is the UMVUE of $\theta$.

(d) The $\hat{\theta} = I(X_1 = X_2)$ takes the value out of the range of $\theta$, so it is not reasonable.

(e) The MLE of $p$ is $\hat{p}_{MLE} = (X_1 + X_2)/2$. So the MLE of $\theta$ is $\hat{\theta}_{MLE} = \hat{p}_{MLE}^2 + (1 - \hat{p}_{MLE})^2$.

5. Let $X_1, \ldots, X_n$ be a random sample with probability density function (pdf)

$$f(x; \theta) = \frac{2}{\sqrt{\pi \theta}} \exp\{-\frac{x^2}{\theta}\}, \quad x > 0,$$

where $\theta > 0$.

(a) This is a regular exponential family. The Fisher information number based on $X_1$ is

$$I_1(\theta) = -E_\theta \frac{d^2 \log f(X_1|\theta)}{d\theta^2} = E_\theta \left(\frac{2X_1^2}{\theta^3} - \frac{1}{2\theta^2}\right) = \frac{1}{2\theta^2}.$$

The CRLB for estimating $\theta$ is $\frac{1}{nI_1(\theta)} = \frac{2\theta^2}{n}$.

(b) Let $T = \frac{2\sum_{i=1}^{n} X_i^2}{n}$. Since $E(T) = \theta$, it is unbiased for $\theta$. And $\text{Var}(T) = \frac{2\theta^2}{n}$, which is equal to the bound. So $T$ is UMVUE of $\theta$. (Note $\sum_{i=1}^{n} X_i^2$ is a complete and sufficient statistic for $\theta$.)

(c) It is easy to check that the MLE is $\hat{\theta} = \frac{W}{n}$, where $W = 2\sum_{i=1}^{n} X_i^2$. The likelihood ratio test statistic for $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$ is given by

$$\lambda(\theta_0) = \frac{L(\theta_0)}{L(\hat{\theta})} = n^{-n/2}(W_{on})^{n/2} \exp[(n - W_{on})/2] = g(W_{on}),$$

where $W_{on} = W/\theta_0$ and $g(x) = (x/n)^{n/2} \exp[(n - x)/2], x > 0$. Note that $\lim_{x \to 0^+} = 0 = \lim_{x \to \infty} g(x)$ and $g(\cdot)$ is concave down with a maximum of $g(n) = 1$ at $x = n$. In other words, $g(\cdot)$ is increasing on $(0, n]$ and decreasing on $[n, \infty)$. So for $\lambda \in (0, 1)$, $g(W_{on}) < \lambda$ if and only if $W_{on} < c_1$ or $W_{on} > c_2$, where $0 < c_1 < n < c_2$ are constants satisfying $g(c_1) = g(c_2) = \lambda$ and

$$\alpha = P_{\theta_0}(W_{on} < c_1 \text{ or } W_{on} > c_2) = 1 - P_{\theta_0}(c_1 \leq W_{on} \leq c_2).$$

Under $H_0 : \theta = \theta_0$, $W_{on} \sim \chi_n^2$. Let $\chi_{n,\gamma}^2$ denote the $\gamma$-quantile of a $\chi_n^2$ variable, i.e., $P(\chi_n^2 \leq \chi_{n,\gamma}^2) = \gamma \in (0, 1)$. So we pick $c_1 = \chi_{n,\gamma_1}^2$ and $c_2 = \chi_{n,\gamma_2}^2$, where $\gamma_2 - \gamma_1 = 1 - \alpha$. 


(d) The acceptance region for the size-\(\alpha\) likelihood ratio test of \(H_0 : \theta = \theta_0\) is

\[
A(\theta_0) = \{(X_1, \cdots, X_n) : c_1 \leq 2 \sum_{i=1}^{n} X_i^2 / \theta_0 \leq c_2\}.
\]

By inverting the region, the 100(1 - \(\alpha\))% confidence region of \(\theta\) is given by \(\{\theta > 0; (X_1, \cdots, X_n) \in A(\theta)\}\), which can be expressed as

\[
\{\theta > 0; c_1 \leq 2 \sum_{i=1}^{n} X_i^2 / \theta_0 \leq c_2\} = \left[\frac{2 \sum_{i=1}^{n} X_i^2}{c_2}, \frac{2 \sum_{i=1}^{n} X_i^2}{c_1}\right].
\]

6. Let \((X_i, Y_i), i = 1, \ldots, n\) be a random sample from a (bivariate) uniform distribution defined on a circle centered at (0, 0) with radius \(\theta > 0\). That is, the joint probability density function of \((X_1, Y_1)\) is given by

\[
f(x, y; \theta) = \begin{cases} \frac{1}{\pi \theta^2} & \text{if } \sqrt{x^2 + y^2} < \theta, \\ 0 & \text{otherwise}. \end{cases}
\]

Define \(R_n = \max_{i=1,\ldots,n} \sqrt{X_i^2 + Y_i^2}\).

(a) For \(0 \leq r \leq \theta\),

\[
P_\theta(R_n \leq r) = \prod_{i=1}^{n} (X_i^2 + Y_i^2 \leq r^2) = (r/\theta)^{2n}.
\]

(b) The variable \(R_n\) has a probability density function given by

\[
f(r) = 2n \theta^{-2n} r^{2n-1}, \quad 0 \leq r \leq \theta.
\]

The mean and second moment of \(R_n\) are then

\[
E_\theta R_n = 2n \theta^{-2n} \int_0^\theta r^{2n} dr = \frac{2n}{2n + 1} \theta,
\]

and

\[
E_\theta R_n^2 = 2n \theta^{-2n} \int_0^\theta r^{2n+1} dr = \frac{2n}{2n + 2} \theta^2.
\]

So the bias of \(R_n\) converges to zero, as \(E_\theta R_n - \theta = -\frac{1}{2n + 1} \theta \to 0\) as \(n \to \infty\). Also, \(E_\theta R_n^2 \to \theta^2\) as \(n \to \infty\), so that the variance of \(R_n\) converges to zero; \(E_\theta R_n^2 - [E_\theta R_n]^2 \to \theta^2 - \theta^2 = 0\). Therefore, \(R_n\) is consistent for \(\theta\).

(c) For any \(\theta > 0\), we have

\[
P(\theta \in I_n) = P(\theta < R_n^{51/(2n+\sqrt{n})}) = P(\theta^{5-1/(2n+\sqrt{n})} \leq R_n) = 1 - P(R_n < \theta^{5-1/(2n+\sqrt{n})}) = 1 - 5^{-2n/(2n+\sqrt{n})} \to 1 - 5^{-1} = 4/5,
\]

as \(n \to \infty\).