Statistics GIDP  
Ph.D. Qualifying Exam  
Theory  
Jan , 2018, 9:00am-1:00pm

Instructions: Provide your answers on the supplied pads of paper; write on only one side of each sheet. Please complete exactly 5 of the 6 problems. Each problem is EQUALLY weighted. Turn in only those sheets you wish to grade. Please put your code ONLY instead of your name on the answer sheets. Stay calm and do your best. Good luck with the exam.

1. $X_1, \ldots, X_n$ are IID from Uniform$(0, 1)$, and $I_n = [0, \frac{1}{n}]$ is a small interval. Define $Y_n$ as the count of the set $\{i : X_i \in I_n\}$.

(a) What is the distribution of $Y_n$?
(b) What is the limit distribution of $Y_n$ as $n \to \infty$? Verify it.

2. Let $X_1, X_2, X_3$ be three random variables from Uniform$(\theta, 3\theta)$, where $\theta > 0$.

(a) Find the constants $a$ and $b$ such that $E[aX(1)] = E[bX(3)] = \theta$.
(b) Calculate Cov$(X(1), X(3))$. (Recall the joint pdf of $X(i)$ and $X(j)$ for a random sample of $X_1, X_2, \ldots, X_n$ is

$$f_{X(i), X(j)}(u, v) = \frac{n!}{(i-1)!(j-1-i)!(n-j)!} f(u) f(v) \times [F(u)]^{i-1}[F(v) - F(u)]^{j-1-i}[1 - F(v)]^{n-j},$$

where $f(\cdot)$ and $F(\cdot)$ are pdf and cdf of $X_i$’s respectively.)
(c) Describe the set $S = \{(c, d) : E(cX(1) + dX(3)) = \theta\}$.
(d) Find $(c, d) \in S$ such that $(c, d)$ minimizes $\text{Var}(cX(1) + dX(3))$ among all $(c, d) \in S$.

3. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Assume that $n = 2k$. Define $U = \frac{1}{2k} \sum_{i=1}^{k} (X_{2i} - X_{2i-1})^2$ and $V = \frac{1}{2n} \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2$.

(a) Show that $U$ converges to $\sigma^2$ in probability.
(b) Show that $V$ converges to $\sigma^2$ in probability.

4. Let $X_1, \ldots, X_n$ be a random sample from Poisson$(\lambda)$.

(a) Show that both $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ are unbiased estimators for $\lambda$.
(b) Which estimator is more efficient for estimating $\lambda$, between $\bar{X}$ and $S^2$? Justify your answer.
(c) Calculate the Cramér-Rao lower variance bound for unbiased estimators of $\lambda^2$.

(d) Is there a UMP test for the hypothesis $H_0 : \lambda < \lambda_0$ vs $H_1 : \lambda > \lambda_0$ at level $\alpha$, where $\alpha \in (0, 1)$? If so, provide the rejection region of the test.

(e) Suppose $\lambda$ has an exponential prior distribution with mean $\theta > 0$. Provide the posterior mean of $\lambda$.

(f) Show that the posterior mean of $\lambda$ obtained in part (e) is consistent for $\lambda$ in the frequentist sense as $n \to \infty$.

5. Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) observations from $\text{Unif}[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$ with probability density function

$$f(x, y; \theta) = \begin{cases} 1 & \text{if } \theta - \frac{1}{2} \leq x \leq \theta + \frac{1}{2}, \\ 0 & \text{otherwise}, \end{cases}$$

where $-\infty < \theta < \infty$ is an unknown parameter. Consider two estimators

$$\hat{\theta}_1 = \frac{X_{(1)} + X_{(n)}}{2}, \quad \hat{\theta}_2 = X_{(n)} - \frac{1}{2},$$

where $X_{(1)}$ and $X_{(n)}$ denote the minimum and maximum order statistics.

(a) Identify a minimal sufficient statistic for $\theta$, and show that it is indeed minimal.

(b) Is the minimal sufficient statistic complete? Prove your answer.

(c) Show that both $\hat{\theta}_1$ and $\hat{\theta}_2$ are maximum likelihood estimators of $\theta$.

(d) Is $\hat{\theta}_2$ unbiased for $\theta$? Explain your answer.

6. Let $(X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n)$ be independently normally distributed random variables, all with the same variance $\theta$ but with $EX_i = EY_i = \mu_i, i = 1, \ldots, n$, all possibly different. The unknown parameter $\theta > 0$ is of major interest for inference, and the unknown parameters $\mu_i, i = 1, \ldots, n$ are considered nuisance parameters.

(a) Obtain the maximum likelihood estimator (MLE) of $\theta$. Denote the MLE by $\hat{\theta}$.

(b) Show that the MLE $\hat{\theta}$ is inconsistent for $\theta$.

(c) Use $\hat{\theta}$ to construct a consistent estimator for $\theta$.

(d) Use $\hat{\theta}$ to construct a $100(1 - \alpha)\%$ confidence interval for $\theta$. 

2
Solutions:

1. (a) Observe that $X_n \sim Bin(n, p_n)$ where $p_n = 1/n$.
   (b) Recall that MGF of $Y_n$ is $[\frac{1}{n} e^t + 1 - \frac{1}{n}]^n$, which converges to $\exp(e^t - 1)$ as $n \to \infty$. i.e. Poisson(1).

2. (a) We can find the pdf of $X_{(1)}$ is $f(t) = \frac{3(\theta - t)^2}{8\theta^3}$. By definition we have $E[X_{(1)}] = \frac{3\theta}{2}$. Similarly pdf of $X_{(3)}$ is $\frac{3(t-\theta)^2}{8\theta^3}$ and $E[X_{(3)}] = \frac{5\theta}{2}$. Hence $a = \frac{2}{3}, b = \frac{2}{5}$.
   (b) The joint pdf of $X_{(1)}$ and $X_{(3)}$ is
   \[
   f_{X_{(1)},X_{(3)}}(u,v) = \frac{3}{4\theta^3}(v-u), \quad \theta < u < v < 3\theta.
   \]
   Then we can get $\text{Cov}(X_{(1)}, X_{(3)}) = \frac{12\theta^2}{5} - \frac{35}{2} \theta^2 = \frac{\theta^2}{20}$.
   (c) $\frac{3}{2}c + \frac{5}{2}d = 1$.
   (d) $\text{Var}[X_{(1)}] = \frac{12\theta^2}{5} - \frac{9}{4} \theta^2 = \frac{32\theta^2}{5} - \frac{25}{4} \theta^2 = \frac{32\theta^2}{5}$. \[
   \text{Var}[cX_{(1)} + dX_{(3)}] = \frac{\theta^2}{20}(3c^2 + 3d^2 + 2cd)
   \]
   subject to $3c + 5d = 2$, we can get the solution $c = 1/9, d = 1/3$.

3. (a) Observe that $X_{2i} - X_{2i-1}$ are i.i.d. for $i = 1, \ldots, k$. By LLN $2U \xrightarrow{P} E(X_2 - X_1)^2 = 2\sigma^2$, therefore, $U \xrightarrow{P} \sigma^2$.
   (b) Define $T = \frac{1}{2k} \sum_{i=1}^{k-1} (X_{2i+1} - X_{2i})^2$. By the same reasoning of $U$’s convergence, we know $T \xrightarrow{P} \sigma^2$. And $V = (U + T)/2$, hence $V \xrightarrow{P} \sigma^2$.

4. Let $X_1, \ldots, X_n$ be a random sample from Poisson($\lambda$).
   (a) $E(\bar{X}) = E(X_1) = \lambda$ and
   \[
   E(S^2) = \frac{1}{n-1} E\left(\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right) = \frac{1}{n-1} [n(\lambda^2 + \lambda) - n(\lambda^2 + \lambda)] = \lambda.
   \]
   Therefore, both estimators are unbiased estimators for $\lambda$.
   (b) $\bar{X}$ is more efficient. We can show that $\bar{X}$ is the UMVUE, as it is unbiased for $\lambda$ and also a function of complete and sufficient statistic $\sum_{i=1}^{n} X_i$.
   (c) Since the Fisher information matrix $I_n(\lambda) = \frac{n}{\lambda}$. The Cramér-Rao bound for unbiased estimators of $\lambda^2$ is $(2\lambda)^2 \cdot \lambda/n = 4\lambda^3/n$.
   (d) The sufficient statistic $T = \sum_{i=1}^{n} X_i$ follows Poisson($n\lambda$), which has the monotone likelihood ratio property. By Karlin Rubin theorem, the UMP test for the hypothesis $H_0 : \lambda < \lambda_0$ vs $H_1 : \lambda > \lambda_0$ at level $\alpha$ has the rejection region $T > t_0$, where $t_0$ satisfies that $\alpha = P(T \geq t_0 | \lambda = \lambda_0)$. 


(e) The posterior distribution of $\lambda$ is

$$
\pi(\lambda | \theta) \propto e^{-n\lambda} \sum_{i=1}^n x_i \frac{1}{\theta} e^{-\lambda/\theta} \propto e^{-(n+\theta/\theta^2)\lambda} \sum_{i=1}^n x_i,
$$

which is $\text{Gamma}(\sum_{i=1}^n x_i + 1, \frac{\theta}{n\theta + 1})$. The posterior mean of $\lambda$ is then

$$
\frac{\theta (\sum_{i=1}^n x_i + 1)}{n\theta + 1} = \frac{n\bar{x} + 1}{n + 1/\theta} = \frac{\bar{x} + 1/n}{1 + 1/(n\theta)}
$$

(f) Yes, the posterior mean of $\lambda$ obtained in part (e) is consistent for $\lambda$ in the frequentist sense, since it converges to $\lambda_{SS}$ as $n \to \infty$.

5. Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) observations from $\text{Unif}[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$ with probability density function

$$
f(x, y; \theta) = \begin{cases} 
1 & \text{if } \theta - \frac{1}{2} \leq x \leq \theta + \frac{1}{2}, \\
0 & \text{otherwise},
\end{cases}
$$

where $-\infty < \theta < \infty$ is an unknown parameter.

(a) By Lehmann-Scheffe, $(X_{(1)}, X_{(n)})$ is minimal sufficient.

(b) It is not complete, since $X_{(n)} - X_{(1)}$ is ancillary.

(c) The likelihood is equal to 1 over $\theta \in [X_{(n)} - \frac{1}{2}, X_{(1)} + \frac{1}{2}]$, thus $L(\hat{\theta}_1) = L(\hat{\theta}_2) = 1$, so both are MLE.

(d) Define $Y_i = X_i - \theta$ for $i = 1, \ldots, n$. Then $Y_i$ is iid $\text{Unif}[-\frac{1}{2}, \frac{1}{2}]$. By symmetry, $Y_{(1)}$ and $-Y_{(n)}$ have the same distribution. So

$$
E(\hat{\theta}_1 - \theta) = E\left(\frac{X_{(1)} - \theta}{2} + \frac{X_{(n)} - \theta}{2}\right) = \frac{1}{2} EY_{(1)} + \frac{1}{2} Y_{(n)} = 0,
$$

so $\hat{\theta}_1$ is unbiased. It is clear that $\hat{\theta}_2 < \theta$, so it is not unbiased.

6. Let $(X_1, Y_1, X_2, Y_2, \ldots, X_n, Y_n)$ be independently normally distributed random variables, all with the same variance $\theta$ but with $EX_i = EY_i = \mu_i, i = 1, \ldots, n$, all possibly different. The unknown parameter $\theta > 0$ is of major interest for inference, and the unknown parameters $\mu_i, i = 1, \ldots, n$ are considered nuisance parameters.

(a) The likelihood function

$$
L(\theta, \mu_1, \ldots, \mu_n) = \frac{1}{(2\pi)^n \theta^n} \exp\left\{-\frac{1}{2\theta} \sum_{i=1}^n [(X_i - \mu_i)^2 + (Y_i - \mu_i)^2]\right\},
$$

and the log likelihood function

$$
l(\theta, \mu_1, \ldots, \mu_n) = \log L = -n \log(2\pi) - n \log(\theta) - \frac{1}{2\theta} \sum_{i=1}^n [(X_i - \mu_i)^2 + (Y_i - \mu_i)^2].
$$
Taking the derivative of $l$ with respect to $\theta$ and $\mu_i$’s, we have

$$l'(\theta) = -\frac{n}{\theta} + \frac{1}{2\theta^2} \sum_{i=1}^{n} [(X_i - \mu_i)^2 + (Y_i - \mu_i)^2] = 0,$$

and

$$l'(\mu_i) = \frac{1}{\theta} [2(X_i - \mu_i) + 2(Y_i - \mu_i)], \quad i = 1, \ldots, n.$$

From $l'(\mu_i) = 0$, we obtain

$$\hat{\mu}_i = \frac{X_i + Y_i}{2}, \quad \hat{\theta} = \frac{1}{2n} \sum_{i=1}^{n} [(X_i - \hat{\mu}_i)^2 + (Y_i - \hat{\mu}_i)^2] = \frac{1}{4n} \sum_{i=1}^{n} (X_i - Y_i)^2.$$

(b) Since $X_i - Y_i \sim N(0, 2\theta)$, then $\frac{(X_i - Y_i)^2}{2\theta} \sim \chi_1^2$, we have $E\hat{\theta} = \frac{1}{4} \cdot 2\theta = \frac{\theta}{2}$, so $\hat{\theta}$ is biased for $\theta$, and therefore it is inconsistent for $\theta$.

(c) Since $E(2\hat{\theta}) = \theta$, by WLLN, $2\hat{\theta} = \frac{1}{4n} \sum_{i=1}^{n} (X_i - Y_i)^2$ is consistent for $\theta$.

(d) Since $\frac{8n\hat{\theta}}{\theta} = \sum_{i=1}^{n} \frac{(X_i - Y_i)^2}{2\theta} \sim \chi_n^2$, we have

$$P\left(\frac{\chi_n^2}{n,1-\alpha/2} < \frac{8n\hat{\theta}}{\theta} < \frac{\chi_n^2}{n,\alpha/2}\right) = 1 - \alpha.$$

Therefore, the 100(1 - $\alpha$)% CI for $\theta$ is

$$\left(\frac{\hat{\theta}}{8n\chi_{n,1-\alpha/2}^2}, \frac{\hat{\theta}}{8n\chi_{n,\alpha/2}^2}\right).$$